Supplementary Notes for ELEN 4810 Lecture 3
Fourier Representations

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Disclaimer: These notes are intended to be an accessible introduction to the subject, with no pretense at completeness. In general, you can find more thorough discussions in Oppenheim's book. Please let me know if you find any typos.

Reading suggestions: Oppenheim and Schafer Section 2.6-2.8

After finishing material from Lecture 2, we will motivate our coming journey into the Fourier domain, and briefly review the (continuous time) Fourier series and Fourier transform. Building on the observation that complex exponentials are eigenfunctions of linear time-invariant systems, we discuss the discrete-time Fourier transform, and a few of its simple properties.

1 The Fourier Domain: A Motivation

In this lecture, we will study frequency-domain representations of discrete time signals and systems. To motivate this journey, let us consider a BIBO stable LTI system, with impulse response h. Because h is stable, $\|h\|_{\ell^1} = \sum_k |h[k]| < +\infty$. Let us consider the system output, when the input consists of the complex exponential

$$x[n] = \exp(j\omega n). \tag{1.1}$$

We calculate

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} \exp(j\omega k)h[n-k] \\ &= \exp(j\omega n) \sum_{k=-\infty}^{\infty} \exp\Big(j\omega(k-n)\Big)h[n-k] \\ &= x[n]H(e^{j\omega}), \end{aligned} \tag{1.2}$$

where we *define*

$$H(e^{j\omega}) \doteq \sum_{n=-\infty}^{\infty} h[n] \exp(-j\omega n). \tag{1.3}$$

Notice that because h is absolute summable, this summation converges, and $H(e^{j\omega})$ is well-defined.

Thus, when $x[n] = \exp(j\omega n)$, the output y is a scalar multiple of the input x:

$$y = H(e^{j\omega})\, x. \tag{1.4}$$

In language inspired by linear algebra, x is an *eigenfunction* of the system. In fact, we will see that fairly general families of signals can be expressed as superpositions of complex exponentials. Understanding how the system acts on these eigenfunctions can yield quite a bit of insight into its behavior on "arbitrary" inputs.

2 Review: Fourier Series and the Fourier Transform

Fourier series. Fourier series give a way of expressing a complicated periodic function as a superposition of much simpler periodic functions, such as complex exponentials (or, if the input is real, sinusoids). Suppose that $x(t)$ is a (continuous-time) periodic signal with period T, so

$$x(t) = x(t+T), \qquad \forall\, t. \tag{2.1}$$

For $k \in \mathbb{Z}$, the complex exponentials

$$e_k(t) = \exp\left(j\frac{2\pi}{T}kt\right) \tag{2.2}$$

are also periodic with period T. Moreover, if we define the inner product[1]

$$\langle f, g\rangle = \int_{-T/2}^{T/2} f(t)g^*(t)dt, \tag{2.3}$$

then

$$\langle e_k, e_\ell\rangle = \begin{cases} T & k=\ell \\ 0 & \text{else.}\end{cases} \tag{2.4}$$

So, $\{e_k \mid k \in \mathbb{Z}\}$ is an orthogonal set.

We look for an approximation of our periodic input $x(t)$ in terms of this set. Namely, we are interested in approximations of the form

$$x(t) \approx \sum_{k=-n}^{n} c_k e_k(t). \tag{2.5}$$

[1] The seemingly arbitrary conjugation on g comes from the definition of an inner product on a complex vector space: an inner product must satisfy $\langle f, g\rangle = \langle g, f\rangle^*$ for every f and g. This ensures that for every f, $\langle f, f\rangle = \langle f, f\rangle^*$, and so $\langle f, f\rangle$ is real. In our particular choice of inner product for functions f defined on $[-T/2, T/2]$ – namely, $\langle f, g\rangle = \int_{t=-T/2}^{T/2} f(t)g^*(t)dt$, this ensures that $\langle f, f\rangle = \int_{t=-T/2}^{T/2} |f(t)|^2 dt = \|f\|_{L^2}^2$ is the energy of f over the interval $[-T/2, T/2]$. See also discussion in the previous lecture.

How should we choose the coefficients c_k? To compute c_ℓ, we can take the inner product of both sides with e_ℓ. Because the e_k are orthogonal, we get

$$\begin{aligned} \langle x, e_\ell \rangle &\approx \sum_{k=-n}^{n} c_k \langle e_k, e_\ell \rangle && (2.6) \\ &= T c_\ell. && (2.7) \end{aligned}$$

This strongly suggests taking

$$\begin{aligned} c_\ell &= \frac{1}{T} \langle x, e_\ell \rangle \\ &= \frac{1}{T} \int_{t=-T/2}^{T/2} x(t) \exp\left(-j\frac{2\pi\ell}{T}t\right) \, dt. && (2.8) \end{aligned}$$

It is not too difficult to show that when the c_ℓ are chosen in this manner, we obtain the best possible approximation to $x(t)$ of the form (2.5), in terms of the L^2 norm (energy). Moreover, if the input x is "nice enough", then this approximation becomes accurate when the number of terms is large enough. For example,

Theorem 2.1. *Let $x : \mathbb{R} \to \mathbb{C}$ be continuous, periodic with period T, and piecewise continuously differentiable. Set*

$$c_k = \frac{1}{T} \langle x, e_k \rangle, \tag{2.9}$$

and

$$\hat{x}_n = \sum_{k=-n}^{n} c_k e_k. \tag{2.10}$$

Then for every t,

$$\lim_{n\to\infty} \hat{x}_n(t) = x(t). \tag{2.11}$$

Moreover, the convergence is uniform, *in the sense that*

$$\lim_{n\to\infty} \max_t |\hat{x}_n(t) - x(t)| = 0. \tag{2.12}$$

Thus, if x is nice enough, not only does $\hat{x}_n(t)$ converge to $x(t)$ for every t, we can actually provide a uniform bound on error at a given n, which holds over all t. If x is not so nice – e.g., discontinuous, then this does not happen. A classical example of this is the Gibbs phenomenon in the Fourier representation of a discontinuous function. The Fourier series approximation converges *much* more slowly in the vicinity of a discontinuity. For functions that are "less nice" – for example, including finitely many discontinuities – the Fourier series still converges, in that the energy of the error goes to zero:

Theorem 2.2. *Let $x : \mathbb{R} \to \mathbb{C}$ be bounded, periodic with period T, and Riemann integrable[2] over the interval $-T/2 \le t \le T/2$. Set*

$$\varepsilon_n = \int_{t=-T/2}^{T/2} |x(t) - \hat{x}_n(t)|^2 \, dt. \tag{2.13}$$

Then $\lim_{n\to\infty} \varepsilon_n = 0$.

This is sometimes referred to as "convergence in L^2", and does not imply pointwise convergence.

[2]A bounded function x is Riemann integrable if and only if its set of discontinuities has measure zero.

	Input signal	Frequency domain representation
Fourier Series	Continuous time, periodic $x(t)$	Series $\{c_k \mid k \in \mathbb{Z}\}$
Fourier Transform	Continuous time $x(t)$	Continuous frequency $X(j\Omega)$, $\Omega \in \mathbb{R}$
Discrete Time Fourier Transform	Discrete time $x[n]$	Continuous frequency, periodic $X(e^{j\omega})$
Discrete Fourier Transform	Discrete time finite or periodic $x[n]$	Discrete finite / periodic $X[k]$

Table 1: Fourier representations of various types of signals

The Fourier transform. The Fourier transform is an (audacious) extension of the Fourier series, from periodic functions $x(t)$ to "arbitrary" functions $x : \mathbb{R} \to \mathbb{C}$. Since x is defined over the entire real line (and is not periodic), we extend our inner product to the entire real line, by writing

$$\langle f, g \rangle = \int_{t=-\infty}^{\infty} f(t) g^*(t)\, dt, \tag{2.14}$$

when f and g satisfy appropriate conditions to ensure that this integral exists.

Rather than using a countable collection of basis functions e_k, corresponding to frequencies $\Omega = 2\pi k/T$, we allow the frequency to be arbitrary, and write

$$\psi_\Omega(t) = \exp(j\Omega t). \tag{2.15}$$

We set

$$\begin{aligned} X(j\Omega) &= \langle x, \psi_\Omega \rangle \\ &= \int_{t=-\infty}^{\infty} x(t) \exp(-j\Omega t)\, dt. \end{aligned} \tag{2.16}$$

This is the *Fourier transform* of x. Whether it even exists depends on the properties of the input x. For example, if x has finite L^1 norm:

$$\int_{t=-\infty}^{\infty} |x(t)|\, dt \;<\; +\infty, \tag{2.17}$$

then the Fourier transform $X(j\Omega)$ exists.

As with the Fourier series, under certain assumptions on x, it is possible to reconstruct x using a superposition of the functions ψ_Ω. The reconstruction formula is

$$x(t) \;=\; \frac{1}{2\pi} \int_{\Omega=-\infty}^{\infty} X(j\Omega) \exp(j\Omega t)\, d\Omega. \tag{2.18}$$

For example, if x is continuous, then the above equation holds for every t. If both $x(t)$ and $X(j\Omega)$ have finite L^1 norm, then it holds for "almost all" t.

The picture. Table 1 shows describes the Fourier domain representations of various types of signals. Our study of discrete time signals will lead us to consider in some depth the *discrete time Fourier transform*, which gives a frequency domain representation of a discrete time signal $x[n]$.

3 The Discrete-Time Fourier Transform

Above, we saw that if we took a stable linear, time-invariant system $\mathcal{T}$ with impulse response $h[n]$ (necessarily satisfying $\sum_{k=-\infty}^{\infty} |h[k]| < +\infty$), and applied it to a complex exponential input

$$x[n] = \exp(j\omega n), \tag{3.1}$$

then the output

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} \exp(j\omega n)\exp\Big(j\omega(k-n)\Big)h[n-k] \\ &= \exp(j\omega n)\underbrace{\sum_{k=-\infty}^{\infty} \exp\Big(-j\omega k\Big)h[k]}_{\doteq H(e^{j\omega})} \\ &= \exp(j\omega n)\,H(e^{j\omega}) \\ &= x[n]\,H(e^{j\omega}). \end{aligned} \tag{3.2}$$

is a scalar multiple $H(e^{j\omega})$ of the input $x[n]$. The function $H(e^{j\omega})$ is tremendously useful for understanding the properties of the system.

The Discrete Time Fourier Transform for absolute-summable $x[n]$

Motivated by the above example, we introduce a *Discrete-Time Fourier Transform* (DTFT). This transform takes a discrete-time signal $x[n]$, and produces a "frequency domain" representation, which we denote $X(e^{j\omega})$. Here, $\omega \in \mathbb{R}$ is a continuous (i.e., real-valued) frequency; for each frequency $\omega \in \mathbb{R}$, we have one complex number $X(e^{j\omega}) \in \mathbb{C}$. We sometimes write the relationship between x and X as

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega}). \tag{3.3}$$

The DTFT $X(e^{j\omega})$ is a function of frequency ω. It may seem strange to write it as $X(e^{j\omega})$ instead of the usual notation for a function of ω (e.g., $g(\omega)$). The reason for the notation $X(e^{j\omega})$ will become clear in about a month when we study the $\mathcal{Z}$-transform; for now, just remember that $X(e^{j\omega})$ is a function of ω.

Definition. Let $x[n]$ be a signal, with

$$\sum_{n=-\infty}^{\infty} |x[n]| < +\infty. \tag{3.4}$$

The *Discrete-Time Fourier Transform* (*DTFT*) of x is a function $X(e^{j\omega})$ of frequency $\omega \in \mathbb{R}$, defined by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]\exp(-j\omega n). \tag{3.5}$$

Notice that when x is absolute summable (i.e., (3.4) holds), the sequence $\tilde{x}[n] = x[n]\exp(-j\omega n)$ is also absolute summable, and so the summation defined in (3.5) converges, and $X(e^{j\omega})$ is well-defined for every $\omega \in \mathbb{R}$.[3]

Periodicity. In the Lecture 2 notes, we discussed some curious properties of discrete-time sinusoids and complex exponentials. The most important fact was that if $L \in \mathbb{Z}$ is an integer, the complex exponentials

$$q[n] = \exp(-j\omega n)$$

and

$$\begin{aligned} q'[n] &= \exp(-j(\omega + 2\pi L)n) \\ &= \exp(-j\omega n)\exp(-j2\pi n) \\ &= \exp(-j\omega n) \\ &= q[n] \end{aligned}$$

are exactly the same sequence! This means that

$$\begin{aligned} X(e^{j(\omega+2\pi L)}) &= \sum_{n=-\infty}^{\infty} x[n]\exp\{-j(\omega + 2\pi L)n\} \\ &= \sum_{n=-\infty}^{\infty} x[n]\exp\{-j\omega n\} \\ &= X(e^{j\omega}). \end{aligned} \tag{3.6}$$

In particular, for all ω, $X(e^{j\omega}) = X(e^{j(\omega+2\pi)})$:

Proposition 3.1 (Periodicity of the DTFT)**.** *The DTFT $X(e^{j\omega})$ is a 2π-periodic function of ω.*

This means that if we know the value of the $X(e^{j\omega})$ over some interval of length 2π, we know the value of $X(e^{j\omega})$ over all ω. We therefore *typically restrict our attention to the interval $-\pi < \omega \leq \pi$.* Occasionally, we may work with the interval $0 \leq \omega < 2\pi$ instead.

The Inverse DTFT. If x is absolutely summable, its discrete time Fourier transform $X(e^{j\omega})$ exists and is continuous. In this situation, we can recover $x[n]$ from $X(e^{j\omega})$ via the inversion formula

$$x[n] = \frac{1}{2\pi}\int_{\omega=-\pi}^{\pi} X(e^{j\omega})\exp(j\omega n)\,d\omega. \tag{3.7}$$

This integral is called the *Inverse Discrete Time Fourier Transform*, and allows us to move from the Fourier domain back to the time domain. You can think of this expression as representing the sequence $x[n]$ as a superposition of complex exponential "basis functions" of the form $\exp(j\omega n)$.
We prove the relationship (3.7) below:

[3] In fact, when x is absolute summable, slightly more can be said: it is actually possible to prove that the partial sums in (3.5) converge uniformly, and that the discrete time Fourier transform $X(e^{j\omega})$ is continuous in ω. We prove this in Theorem A.1 of the appendix of these notes.

Theorem 3.2. *Suppose that*

$$\sum_{n=-\infty}^{\infty} |x[n]| < +\infty. \tag{3.8}$$

Then for every $n \in \mathbb{Z}$*,*

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(e^{j\omega}) \exp(j\omega n)\, d\omega. \tag{3.9}$$

Proof. We first note that for any integer k,

$$\frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} \exp(j\omega k)\, d\omega \;=\; \delta[k]. \tag{3.10}$$

Now, write

$$\begin{aligned}
\frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(e^{j\omega}) \exp(j\omega n)\, d\omega \;&=\; \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} \left(\sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega k) \right) \exp(j\omega n)\, d\omega \\
&= \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} \sum_{k=-\infty}^{\infty} x[k] \exp\bigl\{ j\omega\, (n-k) \bigr\}\, d\omega \qquad \text{(def'n of DTFT)} \\
&= \sum_{k=-\infty}^{\infty} x[k] \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} \exp\bigl\{ j\omega (n-k) \bigr\}\, d\omega \qquad \text{(dominated convergence)} \\
&= \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \qquad \text{(by (3.10))} \\
&= x[n].
\end{aligned} \tag{3.11}$$

In applying the dominated convergence theorem[4], we have used that the partial sums of

$$\sum_{k=-\infty}^{\infty} x[k] \exp\left\{ -j\omega(n-k) \right\}$$

are all dominated by the integrable (uniform) function $g(\omega) = \sum_{k=-\infty}^{\infty} |x[k]|$. □

The DTFT and inverse DTFT allow us to move back and forth between the time and frequency domains in a relatively straightforward way.

Examples. To illustrate the idea, we compute the DTFT of several simple example signals $x[n]$. The DTFT $X(e^{j\omega})$ is complex-valued; typically, we plot it in polar form, by plotting the *magnitude spectrum* $|X(e^{j\omega})|$ and *phase spectrum* $\angle X(e^{j\omega})$ individually. Several examples follow below.

[4] If your background in analysis is such that the phrase "dominated convergence theorem" is unfamiliar, the issue in a nutshell is as follows. In the proof, we interchanged integration over ω and summation over k. This is only possible under specific assumptions on the function being integrated. Fortunately, when $\sum_n |x[n]|$ is finite, these conditions are met, and the proof proceeds without major difficulty. It is very much possible to appreciate the discrete time Fourier transform as an engineering tool without a detailed understanding of the analytical issues associated with its definition and proof. For further analytical work (beyond what will be covered in this class), a detailed understanding is essential.

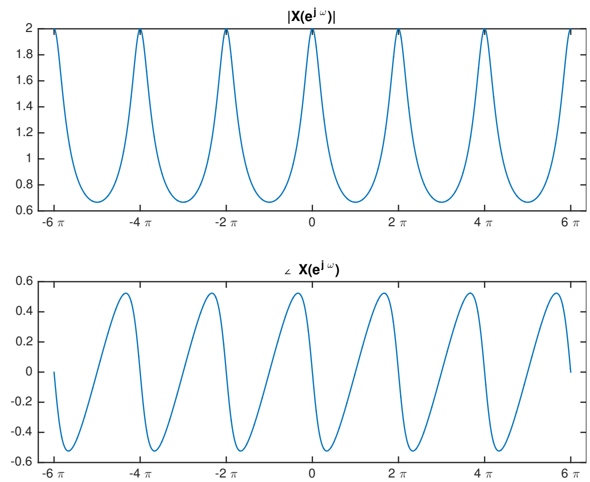

Figure 1: DTFT of the one-sided exponential sequence $(1/2)^n u[n]$.

- **The unit impulse.** If $x[n] = \delta[n]$, $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n] e^{-j\omega n} = 1$, for every ω. Thus, the DTFT of the unit impulse is the constant function $X(e^{j\omega}) = 1$. The magnitude spectrum is $|X(e^{j\omega})| = 1$; the phase spectrum is $\angle X(e^{j\omega}) = 0$.

- **One sided exponential.** If $x[n] = \alpha^n u[n]$, x is absolute summable for $|\alpha| < 1$. Then $X(e^{j\omega})| = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-\alpha e^{-j\omega}}$. The magnitude and phase spectrum are plotted in Figure 1 for the special case $\alpha = 1/2$.

- **Discrete-time box.** If $x[n] = \begin{cases} 1 & 0 \leq n \leq L-1, \\ 0 & \text{else} \end{cases}$, then $X(e^{j\omega}) = \begin{cases} \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} & \omega \neq 0 \\ L & \omega = 0 \end{cases}$. The magnitude and phase spectra are plotted in Figure 2 for the case $L = 10$.

The text contains large tables of DTFT's for additional example signals.

The Discrete Time Fourier Transform for $x[n]$ that are not absolute-summable

Above, we proved that the discrete time Fourier transform is well-defined and its inverse function is valid, for any signal x whose magnitudes are summable. In practice, we will need to work with signals that are not absolute summable. To see why, consider the following example:

An important example. Take some $\omega_c \in (0, \pi)$, and define a function $H_{\text{lp}}(e^{j\omega})$ by

$$H_{\text{lp}}(e^{j\omega}) = \begin{cases} 1 & -\omega_c \leq \omega \leq \omega_c, \\ 0 & \omega_c < |\omega| \leq \pi. \end{cases} \tag{3.12}$$

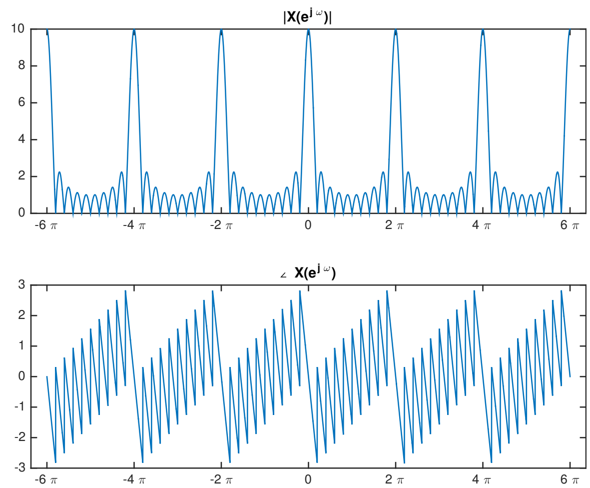

Figure 2: DTFT of the box sequence $x[n] = 1,\ 0 \leq n \leq 9, \quad x[n] = 0$ else

This is the frequency response of an *ideal low-pass filter*, with *cutoff* ω_c. Let us take this specification, and formally push it through the inverse DTFT expression to obtain a sequence $h_{\text{lp}}[n]$:

$$\begin{aligned} h_{\text{lp}}[n] &\doteq \frac{1}{2\pi}\int_{-\pi}^{\pi} H_{\text{lp}}(e^{j\omega})e^{j\omega n}d\omega \\ &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega \\ &= \begin{cases} \omega_c/\pi & n = 0 \\ \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2\pi j n} & \text{else} \end{cases} \\ &= \begin{cases} \omega_c/\pi & n = 0, \\ \frac{\sin(\omega_c n)}{n\pi} & n \neq 0. \end{cases} \end{aligned} \tag{3.13}$$

The entries of $h_{\text{lp}}[n]$ decay to zero as $|n| \to \infty$, but they decay slowly – the n-th term is approximately proportional to $1/n$. It is not too difficult to show that $\|h_{\text{lp}}\|_{\ell^1} = +\infty$ – *the ideal lowpass filter is not absolute summable*. We certainly want to be able to work with a low-pass filter in frequency domain! To do so, we will need to extend the class of signals we can handle.

Although $h_{\text{lp}}[n]$ is not absolute summable, it is *square summable*:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} \left|h_{\text{lp}}[n]\right|^2 &= \frac{\omega_c}{\pi} + \sum_{n=1}^{\infty} \frac{\sin^2(\omega_c n)}{n^2\pi^2} + \sum_{n=-\infty}^{-1} \frac{\sin^2(\omega_c n)}{n^2\pi^2} \\
&= \frac{\omega_c}{\pi} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin^2(\omega_c n)}{n^2} \\
&\leq \frac{\omega_c}{\pi} + \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} \\
&= \frac{\omega_c}{\pi} + 1/3 \quad < \quad +\infty, \qquad (3.14)
\end{aligned}$$

where in the final line we've used that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$, a famous result of Euler. For *square summable* sequences such as $h_{\text{lp}}[n]$, we *can* apply the discrete-time Fourier transform. However, we have to be careful about what to expect. To get some intuition, in Figure 3, we plot the partial sums

$$H_N(e^{j\omega}) \doteq \sum_{n=-N}^{N} h_{\text{lp}}[n] e^{-j\omega n} \qquad (3.15)$$

You can see that away from the discontinuity in H_{lp}, $H_N(e^{j\omega})$ appears to converge to $H_{\text{lp}}(e^{j\omega})$. However, at $\omega = \pm\omega_c$, it does not converge. Near ω_c, we observe ripples, characteristic of the Gibbs phenomenon. In Figure 3, we plot the approximation error

$$\int_{\omega=-\pi}^{\pi} \left|H_N(e^{j\omega}) - H_{\text{lp}}(e^{j\omega})\right|^2 d\omega, \qquad (3.16)$$

as a function of N. You can observe that the approximation error *does* converge to zero as N increases.

Square summable signals. The behavior of the DTFT of the low-pass filter is representative of how the DTFT behaves for general signals $x[n]$ satisfying

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < +\infty, \qquad (3.17)$$

i.e., *square summable* signals. Every absolute summable sequence is square summable. However, some square summable sequences are not absolute summable – $h_{\text{lp}}[n]$ described above is one example.

For sequences that are not absolute summable, the DTFT may not be well-defined for all ω. We saw this above for the ideal lowpass filter. *Another simple example is $x[n] = (1/n)u[n-1]\exp(j\omega_0 n)$. What is the value of $X(e^{j\omega_0})$?*

Although the DTFT of a sequence which is not absolute summable may not be well-defined for all ω, if the sequence is square summable, the DTFT is always well-defined "in the sense of energy:" if we let

$$X_k(e^{j\omega}) = \sum_{n=-k}^{k} x[n] \exp\{-j\omega n\} \qquad (3.18)$$

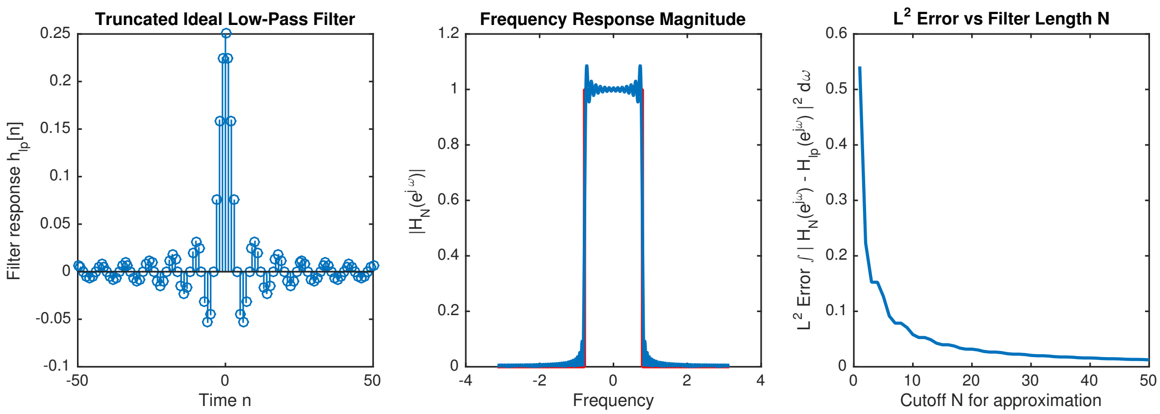

Figure 3: **Truncating the ideal low-pass filter.** Left: ideal lowpass filter $h_{lp}[n]$, restricted to $n = -N, \ldots, N$. Center: magnitude of Fourier transform $|H_N(e^{j\omega})|$ for the truncated lowpass filter. Notice that at the discontinuity $\pm\omega_c$, the magnitude oscillates rapidly. This is an example of the Gibbs phenomenon. Right: L^2 error $\int_{\omega=-\pi}^{\pi} \left|H_N(e^{j\omega}) - H(e^{j\omega})\right|^2 d\omega$, as a function of N. Notice that the L^2 error converges to zero as N increases.

then there is a function $X(e^{j\omega})$ satisfying

$$\lim_{k\to\infty} \int_{\omega=-\pi}^{\pi} \left|X_k(e^{j\omega}) - X(e^{j\omega})\right|^2 d\omega = 0. \tag{3.19}$$

Moreover, the inverse DTFT formula remains valid.

I will not prove either of these facts in lecture. However, I've provided proofs in the appendix to these notes, in case you'd like to develop a more mathematical understanding of what is going on. For our purposes, it is enough to have a qualitative understanding of the issues – in particular, the fact that when $\|x\|_{\ell^1} = +\infty$, the DTFT summation may not exist for certain ω, and that this lack of convergence shows up in finite approximations to the DTFT as a Gibbs-like phenomenon. This fact will have important practical implications when we try to design filters, or study signals (such as musical notes) whose frequency content changes with time.

DTFT for certain useful sequences that are not even square-summable. We can go one step further, and extend the transform to certain signals $x[n]$ that are not even square summable. For example, we could try to take the DTFT of the constant signal $x[n] = 1$, or of a complex exponential $x[n] = \exp(j\omega_0 n)$. In this situation, the summation defined in the DTFT does not converge. Nevertheless, we find it useful to make a *formal* definition of the DTFT for these and other specific cases. For $x[n] = \exp(j\omega_0 n)$, we *define* the Fourier transform to be

$$X(e^{j\omega}) = 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - \omega_0 + 2\pi k), \tag{3.20}$$

where $\delta(\cdot)$ is the (continuous) Dirac delta. This just takes a scaled Dirac delta $2\pi\delta(\omega - \omega_0)$, and extends it to be 2π-periodic.

For our purposes, the statement $\mathrm{DTFT}[\exp(j\omega_0 n)] = 2\pi\sum_{k=-\infty}^{\infty}\delta(\omega-\omega_0+2\pi k)$ is not a mathematical claim – it is a definition. However, it happens to work out, in the sense that if we formally apply the inverse DTFT to $X(e^{j\omega})$, we recover

$$\begin{aligned} x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\sum_k \delta(\omega-\omega_0+2\pi k)\exp(j\omega n)d\omega \\ &= \exp(j\omega n), \end{aligned} \tag{3.21}$$

as desired. Really making this mathematically rigorous requires us to delve into the theory of generalized functions, which is substantially beyond the scope of this course. For our purposes we simply accept (3.20) as a definition.

Using the relationships

$$\cos(\omega_0 n) = \frac{\exp(j\omega_0 n)+\exp(-j\omega_0 n)}{2}, \qquad \sin(\omega_0 n) = \frac{\exp(j\omega_0 n)-\exp(-j\omega_0 n)}{2j}, \tag{3.22}$$

and formal manipulations, we can also obtain formal expressions for the DTFT of $x[n] = \cos(\omega_0 n)$ and $x[n] = \sin(\omega_0 n)$.

4 Symmetry properties of the DTFT

The most basic symmetry property of the DTFT states that conjugation in time domain is equivalent to a conjugation and flip about zero in the frequency domain:

Proposition 4.1. *Let x be a signal with DTFT $X(e^{j\omega})$. Then the DTFT of the complex conjugate signal $\bar{x} = x^*$ is $\bar{X}(e^{j\omega}) = X^*(e^{-j\omega})$.*

Proof. We simply calculate

$$\begin{aligned} \bar{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} \bar{x}[n]e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} x^*[n]e^{-j\omega n} \\ &= \left(\sum_{n=-\infty}^{\infty} x[n]e^{j\omega n}\right)^* \\ &= X(e^{-j\omega})^*, \end{aligned} \tag{4.1}$$

as claimed. □

This property is useful because it implies several facts about the discrete-time Fourier transform of a real signal $x[n]$. Namely,

Proposition 4.2. *Let $x[n]$ denote a real signal, with discrete-time Fourier transform $X(e^{j\omega})$. Then X is conjugate-symmetric: $X(e^{j\omega}) = X^*(e^{-j\omega})$.*

Proof. If x is real, $x^* = x$. Applying the previous proposition gives the result. □

In particular, this implies that if x is a real signal, the real part of $X(e^{j\omega})$ is even (symmetric), and the imaginary part is odd (antisymmetric). It also implies that the magnitude is even, and the phase is odd.

5 Basic DTFT relationships

Linearity.

Proposition 5.1. *The discrete time Fourier transform is linear: if $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$ are the DTFT's of signals x_1 and x_2, respectively, and $\alpha, \beta \in \mathbb{C}$ are any scalars, then*

$$\alpha X_1(e^{j\omega}) + \beta X_2(e^{j\omega}) \tag{5.1}$$

is the DTFT of the combination $\alpha x_1 + \beta x_2$.

Proof. Exercise: use the definition. □

Time Shifts. The time shift property says that a *shift* in time domain is equivalent to *multiplication by a complex exponential* in frequency domain:

Proposition 5.2. *Let $x[n]$ denote a signal with discrete time Fourier transform $X(e^{j\omega})$, and let $\bar{x}[n] = x[n - n_0]$, for some $n_0 \in \mathbb{Z}$. Then the DTFT $\bar{X}(e^{j\omega})$ of $\bar{x}$ is given by*

$$\bar{X}(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega}). \tag{5.2}$$

Proof. Calculate

$$\begin{aligned} \bar{X}(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n - n_0] e^{-j\omega n} && (5.3) \\ &= e^{-j\omega n_0} \sum_{n=-\infty}^{\infty} x[n - n_0] e^{-j\omega (n - n_0)} && (5.4) \\ &= e^{-j\omega n_0} X(e^{j\omega}), && (5.5) \end{aligned}$$

as claimed. □

Convolution in time.

Theorem 5.3. *Suppose that $y = x * h$, where x and h are absolute summable, with discrete time Fourier transforms $X(e^{j\omega})$ and $H(e^{j\omega})$, respectively. Then the discrete time Fourier transform $Y(e^{j\omega})$ of Y satisfies*

$$Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}). \tag{5.6}$$

Proof. We calculate

$$\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x[k]h[n-k] \right) e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega k) h[n-k] \exp(-j\omega(n-k)) \\
&= \sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega k) \sum_{n=-\infty}^{\infty} h[n-k] \exp(-j\omega(n-k)) \\
&= H(e^{j\omega}) \sum_{k=-\infty}^{\infty} x[k] \exp(-j\omega k) \\
&= H(e^{j\omega}) X(e^{j\omega}),
\end{aligned} \tag{5.7}$$

as claimed. □

Thus, convolution in time becomes (pointwise) multiplication in the frequency domain.[5] This property is one of the reasons that the DTFT is such a powerful tool for studying linear time invariant systems.

Modulation in time.

Theorem 5.4. *Suppose that $y[n] = x[n]w[n]$, and that $Y(e^{j\omega})$, $X(e^{j\omega})$ and $W(e^{j\omega})$ are the DTFT's of y, x, and w, respectively. Then*

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta. \tag{5.8}$$

Proof. We calculate

$$\begin{aligned}
Y(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} w[n]x[n]e^{-j\omega n} \\
&= \sum_{n=-\infty}^{\infty} w[n] \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) e^{j\theta n} d\theta \; e^{-j\omega n} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) \sum_{n=-\infty}^{\infty} w[n] e^{-j(\omega-\theta)n} \, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta}) W(e^{j(\omega-\theta)}) d\theta,
\end{aligned} \tag{5.9}$$

as claimed. □

[5] This is sometimes verbalized as "the fourier transform diagonalizes convolution".

Thus, multiplication (modulation) in time domain becomes (continuous-time) convolution in the frequency domain.

Inner products and Parseval's theorem. We close our discussion of basic properties of the DTFT with a result showing that the DTFT preserves inner products, in the following sense. If we define, for two square summable sequences x and y,

$$\langle x, y\rangle = \sum_{n=-\infty}^{\infty} x[n]y^*[n], \tag{5.10}$$

and for any two "sufficiently nice" functions $X(e^{j\omega})$ and $Y(e^{j\omega})$,

$$\langle X, Y\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega, \tag{5.11}$$

then whenever X and Y are the DTFT's of x and y, respectively,

$$\langle x, y\rangle = \langle X, Y\rangle . \tag{5.12}$$

That is to say,

Theorem 5.5. *For all square summable x and y with discrete time Fourier transforms $X(e^{j\omega})$ and $Y(e^{j\omega})$,*

$$\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega. \tag{5.13}$$

Proof.

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right)Y^*(e^{j\omega})d\omega && (5.14)\\
&= \sum_{n=-\infty}^{\infty} x[n]\frac{1}{2\pi}\int_{-\pi}^{\pi} Y^*(e^{j\omega})e^{-j\omega n}d\omega && (5.15)\\
&= \sum_{n=-\infty}^{\infty} x[n]\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}d\omega\right)^* && (5.16)\\
&= \sum_{n=-\infty}^{\infty} x[n]y^*[n], && (5.17)
\end{aligned}$$

as desired. □

Applying this result with $y[n] = x[n]$, we obtain an important corollary, known as Parseval's theorem. Parseval's theorem states that the DTFT preserves the energy of the signal, up to a multiplication by $1/2\pi$:

Theorem 5.6. *For any square summable signal x,*

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega. \tag{5.18}$$

Proof. Apply Theorem 5.5 with $y = x$. □

A Optional Appendix: Continuity of the DTFT, theory for square-summable sequences

As in previous lecture notes, in this appendix, we record some results and derivations that may be interesting to the theorists and completists amongst you. This is not required reading, but could enhance your understanding.

Continuity of the DTFT for absolute-summable x

Theorem A.1. *Suppose that $\|x\|_{\ell^1} = \sum_{k=-\infty}^{\infty} |x[k]| < +\infty$. Then $X(e^{j\omega})$ is a continuous function of ω.*

Proof. Because $X(e^{j\omega})$ is 2π periodic, it is enough to prove that $X(e^{j\omega})$ is continuous on the interval $\omega \in [-\pi, \pi]$. We will show that for every $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|\omega - \omega'| < \delta$, $|X(e^{j\omega}) - X(e^{j\omega'})| < \varepsilon$. Because $\|x\|_{\ell^1} < +\infty$, there exists an integer N such that

$$\sum_{|k|>N} |x[k]| < \varepsilon/4. \tag{A.1}$$

Moreover, there exists an η (depending on N) such that for every $\omega, \omega' \in [-\pi, \pi]$ and every integer k of size $|k| \le N$,

$$\left| e^{-j\omega k} - e^{-j\omega' k} \right| \le \eta \left| \omega - \omega' \right|. \tag{A.2}$$

Choose $\delta \le \frac{\varepsilon}{2\eta \|x\|_{\ell^1}}$, so that for all ω, ω' satisfying $|\omega - \omega'| \le \delta$,

$$\begin{aligned}
\left| X(e^{j\omega}) - X(e^{j\omega'}) \right| &= \left| \sum_{k=-\infty}^{\infty} x[k] e^{j\omega k} - \sum_{k=-\infty}^{\infty} x[k] e^{j\omega' k} \right| && \text{(A.3)}\\
&\le \sum_{k=-\infty}^{\infty} |x[k]| \left| e^{j\omega k} - e^{j\omega' k} \right| && \text{(A.4)}\\
&= \sum_{k=-N}^{N} |x[k]| \underbrace{\left| e^{j\omega k} - e^{j\omega' k} \right|}_{\le \eta\delta} + \sum_{|k|>N} |x[k]| \underbrace{\left| e^{j\omega k} - e^{j\omega' k} \right|}_{\le 2.} && \text{(A.5)}\\
&\le \eta\delta \left\| x \right\|_{\ell^1} + 2\varepsilon/4 && \text{(A.6)}\\
&\le \varepsilon, && \text{(A.7)}
\end{aligned}$$

by our choice of δ. $\square$

L^2 convergence of the DTFT for square-summable x

We prove that the DTFT of a square-summable sequence converges in "mean square sense:"

Theorem A.2. *Let $x[n]$ be a sequence, with $\sum_{n=-\infty}^{\infty} |x[n]|^2 < +\infty$. Set $X_N(e^{j\omega}) = \sum_{n=-N}^{N} x[n] e^{-j\omega n}$. Then there exists function $X(e^{j\omega})$ which is square-integrable over $[-\pi, \pi]$, such that*

$$\lim_{N\to\infty} \int_{\omega=-\pi}^{\pi} \left| X_N(e^{j\omega}) - X(e^{j\omega}) \right|^2 d\omega = 0. \tag{A.8}$$

Proof. We first note that for k an integer,

$$\int_{\omega=-\pi}^{\pi} e^{j\omega k} d\omega = \begin{cases} 2\pi & k = 0 \\ 0 & \text{else} \end{cases} \tag{A.9}$$

For functions $f(\omega)$, $g(\omega)$ which are square integrable on $[-\pi, \pi]$, define

$$\langle f, g \rangle = \int_{\omega=-\pi}^{\pi} f(\omega) g^*(\omega) d\omega. \tag{A.10}$$

Write

$$\|f\|_{L^2} = \sqrt{\langle f, f \rangle} = \sqrt{\int_{\omega=-\pi}^{\pi} |f(\omega)|^2 d\omega}. \tag{A.11}$$

Notice that for $N > N'$,

$$\begin{aligned} \left\| X_N(e^{j\omega}) - X_{N'}(e^{j\omega}) \right\|_{L^2}^2 &= \left\| \sum_{N'<|n|\leq N} x[n] e^{-j\omega n} \right\|_{L^2}^2 & \text{(A.12)} \\ &= \left\langle \sum_{N'<|n|\leq N} x[n] e^{-j\omega n}, \sum_{N'<|m|\leq N} x[m] e^{-j\omega m} \right\rangle & \text{(A.13)} \\ &= \sum_{N'<|n|\leq N} \sum_{N'<|m|\leq N} x[n] x^*[m] \left\langle e^{-j\omega n}, e^{-j\omega m} \right\rangle & \text{(A.14)} \\ &= 2\pi \sum_{N'<|n|\leq N} |x[n]|^2 & \text{(A.15)} \\ &\leq 2\pi \sum_{|n|>N'} |x[n]|^2. & \text{(A.16)} \end{aligned}$$

Since $\sum_n |x[n]|^2$ is finite, as $N' \to \infty$, $\sum_{|n|>N'} |x[n]|^2 \to 0$. Hence, the sequence X_N is a Cauchy sequence. Because $L^2([-\pi, \pi])$ is complete, the sequence X_N converges in L^2 to a limit $X(e^{j\omega})$. $\square$